J. Fluid Mech. (1987), *vol.* 178, *pp.* 549–566
Printed in Great Britain

On the stability of a columnar vortex to disturbances with large azimuthal wavenumber: the lower neutral points

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(Received 11 October 1985 and in revised form 29 August 1986)

The inviscid instability of a columnar trailing-line vortex at large values of the azimuthal wavenumber n near neutral conditions is considered. This extends an earlier analysis (Leibovich & Stewartson 1983), which is not accurate near the limiting values of the axial wavenumber for which instabilities exist. Here an asymptotic expansion is derived for the solution in the neighbourhood of the lower neutral point and the results compared with existing computations at moderate values of n. For these weak instabilities disturbances are centred near the axis of the vortex and the relevant equation is solved in the complex plane by a generalized saddle-point method. In addition, the marginal stability of the vortex is examined, and an estimate obtained of the value of the swirl parameter above which the vortex is stable at large values of n.

1. Introduction

A columnar vortex is an axially symmetric swirling fluid motion in which the axial (z-directed) and azimuthal (θ-directed) velocities are functions only of r, the distance from the vortex axis. Here we denote these quantities by $W(r)$ and $V(r)$ respectively, and cylindrical (r, θ, z)-coordinates are implied. The stability characteristics of this vortex when subjected to inviscid disturbances have recently attracted much attention, notably by Lessen, Singh and Paillet (1974), Duck & Foster (1980) and Foster & Duck (1982). These numerical studies describe the growth characteristics of strongly unstable normal modes of the form $\exp[\mathrm{i}(\alpha z - n\theta - \omega t)]$ with moderate azimuthal wavenumbers n and indicate that the growth rate increases with n. Leibovich & Stewartson (1983), in a paper to which we shall subsequently refer as I, extended these studies and in addition developed an asymptotic theory, valid for $n \gg 1$, which demonstrates that the maximum growth rate indeed increases with n and tends to a limit as $n \to \infty$. Moreover, they were able to give a sufficient (but not necessary) condition for the instability of a columnar vortex. If $\Omega = V/r$ and $\Gamma = rV$ are the angular velocity and circulation of the vortex then the flow is unstable to inviscid disturbances if, at any point of the fluid,

$$\Sigma(r) \equiv V\Omega'(\Gamma'\Omega' + W'^2) < 0, \tag{1.1}$$

where primes denote derivatives with respect to r. We shall refer to flows in which (1.1) holds as 'strongly unstable'. The condition

$$\Gamma\Gamma' > \tfrac{1}{8}r^3 W'^2$$

at all points of the flow, was already known to be a sufficient condition for the columnar vortex to be stable to axisymmetric disturbances (Howard & Gupta 1962), but it is not necessary.

The mechanism leading to instabilities characterized by (1.1) has been discussed by Leibovich (1983) (see also Leibovich 1984). For 'pure' vortices having $W \equiv 0$, Rayleigh's parameter is $\Phi = 2\Omega\zeta$, where ζ is the axial vorticity of the basic flow. Pure vortices are stable to axially symmetric perturbations (Rayleigh 1880, 1916) if $\Phi > 0$ and unstable (Synge 1933) if $\Phi < 0$. If $\boldsymbol{e}_w$ is a unit vector parallel to the wavenumber vector, $\boldsymbol{e}_w = (-(n/r)\,\boldsymbol{e}_\theta + \alpha\boldsymbol{e}_z)/(\alpha^2+n^2/r^2)^{\frac{1}{2}}$ where $\boldsymbol{e}_\theta$ and $\boldsymbol{e}_z$ are unit vectors in the θ- and z-directions, then Rayleigh's parameter can be written

$$\Phi = 2(\boldsymbol{\Omega}\cdot\boldsymbol{e}_w)\,(\boldsymbol{\zeta}\cdot\boldsymbol{e}_w), \tag{1.2}$$

where $\boldsymbol{\zeta}$ is the vorticity vector of the basic flow, and $\boldsymbol{\Omega}$ is the angular velocity of fluid particles about the axis. In Rayleigh's case, $\boldsymbol{e}_w$ is along the axis, and the rate-of-strain in the basic ('pure'-vortex) flow in the axial direction ($\boldsymbol{e}_w$) is zero at each r. For flows with $W' \neq 0$, there is also, at each value of r, a principal axis of the rate-of-strain tensor along which the rate-of-strain is zero. This axis is tangent to the cylinder at r, and its direction is given by the unit vector $\boldsymbol{\lambda} = (-W'\boldsymbol{e}_\theta + r\Omega'\boldsymbol{e}_z)/(r^2\Omega'^2 + W'^2)^{\frac{1}{2}}$.

The criterion (1.1) is developed in I by an asymptotic analysis showing that unstable waves are possible (as $|n|$ and $|\alpha| \rightarrow \infty$) for wavenumber ratios n/α only if there exist locations $r = r_0$ in this flow with

$$\frac{n}{\alpha} = \frac{W'(r_0)}{\Omega'(r_0)}, \tag{1.3a}$$

and this condition corresponds to waves, at $r = r_0$, with rays parallel to $\boldsymbol{\lambda}$, the principal axis with zero rate of strain: $\boldsymbol{e}_w = \boldsymbol{\lambda}$ for these waves. They further showed in I that the growth rate is then given by

$$\omega_i = (-\Phi)^{\frac{1}{2}}, \tag{1.3b}$$

with $\boldsymbol{e}_w$ replaced by $\boldsymbol{\lambda}$. Criterion (1.1) is, except for a positive factor, just $\Phi < 0$, and it arose by the steps just outlined. This leads to the following physical interpretation of (1.1): in a curvilinear coordinate system having $\boldsymbol{e}_w$ as one axis the flow appears (since the rate of strain in the $\boldsymbol{\lambda}$-direction vanishes) locally to be a pure vortex at a radius where $\boldsymbol{e}_w = \boldsymbol{\lambda}$, and is centrifugally unstable according to the Rayleigh/Synge criterion. For very large values of n and α, boundary effects exert little influence and local dynamics control behaviour. This interpretation of (1.1) also explains why waves with one screw sense (i.e. sign of n/α) tend to be more unstable (because of (1.3a)) than waves with the opposite screw sense, a fact found by Lessen *et al.* (1974). Ludwieg (1960) also arrived at (1.1) as a stability criterion for flow in a narrow annular gap by physical reasoning. His treatment did not permit its status as either a sufficient or necessary motion for stability to be more determined: indeed it was applied as if both necessary and sufficient. A mathematically derived condition subsequently derived by Ludwieg (1961) for the narrow-gap annulus differs slightly from (1.1) and is not applicable to flows other than in narrow gaps. Very recently, Emanuel (1984) has rediscovered Ludwieg's (1960) derivation of (1.1) and the general growth rate given in equation (5.8) of I.

A vortex which has been studied extensively is a model of Batchelor's (1964) solution for swirling wakes

$$W = \mathrm{e}^{-r^2}, \quad V = \frac{q}{r}(1-\mathrm{e}^{-r^2}). \tag{1.4}$$

These profiles also serve as good analytical fits to experimental data from various experiments on vortices in tubes. For q a constant, which without loss of generality we take to be positive, condition (1.1) implies that the vortex is strongly unstable for $q < \sqrt{2}$ when non-axisymmetric perturbations are permitted.

It is extremely difficult to locate modes that are weakly unstable or neutral by direct numerical computation, though in I a certain amount of progress for the case of the vortex given by (1.4) was reported. To each pair of values of n and α, α being the axial wavenumber of the disturbance, there corresponds a number of different modes and these tend to coalesce near the neutral state. The numerical analyst therefore has to guard against mode-jumping as the parameters of the problem are varied, and a theoretical treatment of the properties of the near-neutral modes is desirable.

The asymptotic analysis of I predicted maximum growth rates and associated wavenumbers very well for strongly unstable flows, but the analysis fails in the immediate neighbourhoods of the neutral points, i.e. the upper and lower bounds of the admissible values of the ratio $\alpha/n (n \neq 0)$ for which unstable modes occur, and we seek here to deal with the case of the lower neutral point. A previous paper (Stewartson & Capell 1985) considers the upper neutral point. The latter, in the limit $n \rightarrow \infty$, is given by $\alpha q \rightarrow n-$ and in this neighbourhood the eigensolutions are, as in I, ring-modes in the sense that the disturbance is centred on a finite non-zero value of r. Both analytic and numerical solutions of the limiting form of the governing equation were found and these were shown to match with those of I as the appropriate variable $n(n-\alpha q)$ became large and positive. No unstable modes with $n(n-\alpha q) < 0$ are expected.

The structure of the modes near the lower neutral point is more subtle. It is shown here that, in the limit $n \rightarrow \infty$, this point is $\alpha = \frac{1}{2}nq$ and that the relevant equation, instead of having a saddle on the real axis of r which led to the ring modes noted above, has the corresponding singularity at a complex value of r and in a neighbourhood $O(n^{-\frac{1}{2}})$ of the origin; at the neutral condition this point lies on the imaginary axis of r. An asymptotic expansion for the growth rate is determined, by contour deformation through this singularity and integration along a steepest descent path, up to and including the term that yields the mode-separation. As expected, the term yielding mode-separation is smaller than that in the more unstable range of axial wavenumber considered in I. For moderate values of n the numerical studies show instability at values of α/n that are considerably less than $\frac{1}{2}q$ and as the asymptotic expansion is of limited practical use unless $\alpha - \frac{1}{2}nq$ is small, a heuristic approximation is, in addition, briefly investigated.

Finally, attention is turned to the marginal stability of the vortex. In the limit $n \rightarrow \infty$ this occurs at $q = \sqrt{2}$ in the neighbourhood of which there are two neutral points; if $q > \sqrt{2}$ they are both of the 'lower'-neutral-point type considered in this paper, but if $q < \sqrt{2}$ there is one of this type but the second resembles the upper neutral point discussed by Stewartson & Capell. In addition, an upper bound on q for instability is obtained: it is shown that, when $n \gg 1$, the vortex is stable if $q > \sqrt{2}(1+n^{-1}/\sqrt{6})$. The results in the neighbourhood of marginal stability show reasonably good agreement with the numerical studies.

2. General analysis near neutral points

We define a cylindrical polar coordinate system (r, θ, z) where z denotes distance parallel to the axis of the columnar vortex measured from some assigned origin, r the distance from this axis and θ is the azimuthal angle. The fluid has a velocity component $W(r)$ parallel to the z-axis and an azimuthal component $V(r)$, where W and V are positive functions of r only. The vortex is taken to be unbounded but as $r \to \infty$, $W \to 0$, while rV tends to a finite limit.

This vortex is now subjected to an infinitesimal disturbance in which each component of velocity and the pressure is assumed to be of the form

$$\Psi(r) \exp[\mathrm{i}\alpha z - \mathrm{i}n\theta - \mathrm{i}\omega t], \tag{2.1}$$

where the Ψ's are functions of r only, α is a given real constant which may be taken to be positive without loss of generality, n is a prescribed integer, and ω is a complex constant to be found. We shall define $\beta = \alpha/n$. Numerical studies by Lessen *et al.* (1974) have demonstrated that if, as we shall assume here, the fluid is inviscid, modes with n positive are likely to be the most unstable. Indeed, for the trailing vortex given by (1.4), they found no unstable modes if $n = -1$ and $q > 0.08$. We therefore consider n to be positive.

If $(1+\beta^2 r^2)^{\frac{1}{2}} r^{-\frac{3}{2}} \phi(r)$ is the radial dependence (u) of the radial component of the velocity perturbation then the equation for u given by Howard & Gupta (1962) is placed in the form (see I)

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}r^2} = n^2 \frac{1+\beta^2 r^2}{r^2} \left\{1 + \frac{a(r)}{n\gamma} + \frac{b(r)}{\gamma^2} - \frac{1+10\beta^2 r^2 - 3\beta^4 r^4}{4n^2(1+\beta^2 r^2)^3}\right\} \phi, \tag{2.2}$$

where

$$\left.\begin{aligned} \gamma(r) &= n[\beta W(r) - \Omega(r)] - \omega \equiv n\Lambda(r) - \omega, \\ a(r) &= r\frac{\mathrm{d}}{\mathrm{d}r}\left\{W'(r)\left[\frac{\beta r^2 + q}{r(1+\beta^2 r^2)}\right]\right\}, \\ b(r) &= -\frac{2\beta\Gamma}{r(1+\beta^2 r^2)}[\beta\Gamma'(r) + W'(r)], \\ \Omega &= \frac{V}{r}, \quad \Gamma = Vr, \quad q = -\frac{\Gamma'(r)}{W'(r)}, \end{aligned}\right\} \tag{2.3}$$

and, as before, primes denote differentiation with respect to r. For the trailing vortex of (1.4) the two definitions of q are equivalent. The associated boundary conditions are

$$\phi \to 0 \quad \text{as} \quad r \to 0, \quad r \to \infty. \tag{2.4}$$

We wish to find the properties of ω for which non-trivial functions ϕ can be found when n is large.

The computations by Lessen *et al.* (1974) suggest that for unstable disturbances the maximum growth rate is achieved in the limit $n \to \infty$. The studies, both numerical and analytical, of I support this conclusion for strongly unstable flows. Two limiting situations were noted in I but not extensively discussed; both of these cases deal with conditions that correspond formally to the neutral limit of strongly unstable flows. One of these has already been considered elsewhere (Stewartson & Capell 1985) and the second is the subject of the present study. To explain, we briefly review the procedure underlying the analysis undertaken in I.

The governing equation (2.2) may be written in the form

$$\phi'' = K(r; n, \beta, \omega)\,\phi. \tag{2.5}$$

Since $K \propto n^2$, for $n \gg 1$ it is a rapidly varying function. The method used in I centres on the existence of a point $r = r_0$ where K is stationary; this is the point appearing in (1.3*a*). In I, such a stationary point was found to exist on the real r-axis, provided the condition (1.1) could be met, and the treatment in I was limited to such cases. To treat properly nearly neutral modes, the same strategy applies, but more accurate calculations need to be made to determine the location of stationary points, and these stationary points need not be restricted to the real r-axis. The contour of integration, which starts at $r = 0$ and extends to $r = \infty$, may be deformed from the real r-axis into the complex r-plane so as to pass through the stationary point. Provided no singularities of $K(r; n, \beta, \omega)$ are enclosed between the real r-axis and the deformed path C, the eigenvalues ω are not affected by the path deformation. Assuming this to be the case, the results required here may be found by a local analysis in the neighbourhood of the stationary point. In the Appendix, the question of path deformation is considered further to confirm that it can be done as presumed (this is, without passing over a singularity of K): this is done by computing steepest descent paths which, though not necessary, have the nice property that the choice of such paths leads to exponentially small corrections to the local analysis.

The location of the stationary point itself depends on ω, of course, and so this procedure is of immediate use in computing ω only in special circumstances, as in the case here of large n. (On the other hand, iterative methods of numerically integrating (2.5) by shooting proceed by successively fixing ω, then integrating; the path deformation used here perhaps could be incorporated into such a process and this may prove advantageous.)

Near $r = r_0$, K takes the form

$$K = K_0 + K_2(r-r_0)^2 + K_3(r-r_0)^3 + \dots, \tag{2.6}$$

where the K_j are independent of r but depend on n. Under the assumption that the K_j are suitably ordered with n, the K_j $(j > 2)$ may be neglected. The entire path of integration can be comprised of an inner region traversing $r = r_0$ and on which K is asymptotically represented by (2.6), and an outer path connecting the inner path to the boundary points $r = 0$ and $r = \infty$. The solution will decay exponentially fast on the outer path provided that it may be chosen as explained in the Appendix. To match with this outer approximation, we must have $\phi \to 0$ as r leaves the immediate neighbourhood of $r = r_0$. On the inner path, ϕ is given by a Weber function, and solutions which decay away from the neighbourhood $r = r_0$ are possible only if

$$K_0 K_2^{-\frac{1}{2}} = -(2s-1), \tag{2.7}$$

where s is a positive integer. Thus, in this first approximation the expansion (2.6) and the condition (2.7) permit the existence of an eigensolution. Since $K \propto n^2$, these considerations imply that, to lowest order K and its first derivative vanish together at $r = r_0$ for the class of eigensolutions postulated. These conditions on K were shown in I to be approximately satisfied for real r_0 when

$$\Lambda'(r) = \beta W'(r) - \Omega'(r) = 0, \tag{2.8}$$

which expresses the vanishing of $K'(r)$, and

$$b(r) + (n\Lambda(r) - \omega)^2 = 0, \tag{2.9}$$

which expresses the vanishing of K itself. Consequently,

$$\begin{aligned}\omega &= n\Lambda(r) \pm \mathrm{i}b^{\frac{1}{2}} \\ &= n\Lambda(r) \pm \mathrm{i}\left(\frac{2\beta\Gamma}{r(1+\beta^2r^2)}\right)^{\frac{1}{2}}(-\beta\Gamma' - W')^{\frac{1}{2}}.\end{aligned} \tag{2.10}$$

It follows, as shown in I, that the vortex is unstable to disturbances with large azimuthal wavenumber if (1.1) holds. The argument fails if $b < 0$ and no conclusions can be reached in this case. It is notable, however, that no unstable modes are known to us that are concentrated about values of r for which $b < 0$.

There are two parameter ranges which merit further study, both corresponding to the (formal) neutral limit of the strongly unstable modes of I. The first arises when the value of β, or of r defined by (2.8), implies that $b(r) \ll 1$, so that $a(r)/n\gamma$ may not be neglected in comparison with $b(r)/\gamma^2$: for the trailing vortex of (1.4) this, when $q < \sqrt{2}$, is the neighbourhood of the upper neutral point and is the subject of the study by Stewartson & Capell (1985) noted above. The second, with which we shall be concerned here, arises when the value of r is small and, because of the relative sizes of the various terms in K, the requirement that $\mathrm{d}K/\mathrm{d}r = 0$ cannot be met by simply setting $\Lambda'(r) = 0$ as in (2.8). This is the situation that prevails in the neighbourhood of the lower neutral point and leads to considerable computational difficulties as experienced by Duck & Foster (1980) and by the present authors in I. There are two reasons for this: firstly γ vanishes at points in the complex plane of r that are very close to the real axis, and secondly the phenomenon of mode-jumping assumes serious proportions. It is clear from (2.7) that when $n \gg 1$ there are a large number of modes at fixed β for which, as shown in I, the separation in ω_i between adjacent modes is $O(b^{\frac{1}{4}}n^{-\frac{1}{2}})$. Thus the possibility of an eigensolution jumping from one mode to an adjacent one, as the investigator attempts to follow its properties when β varies, is always a matter for concern but it becomes serious as $b \to 0$. Indeed, apart from the recent computations of Stewartson & Capell (1985), who adopted an appropriate asymptotic form for the governing equation in the neighbourhood of the upper neutral point, all the computations so far reported had to be terminated before either neutral point was reached. We shall develop here an asymptotic theory, on the lines of but more refined than that of I, which will enable us to distinguish between the various modes near the lower neutral point.

3. The lower neutral points: first stage

Near the lower neutral point the value r_0 of r at which $\mathrm{d}K/\mathrm{d}r = 0$ is small. Anticipating the dominant behaviour of the eigenfunctions to occur near $r = 0$, we first make a preliminary stretching and then treat the simple approximate form of the problem by the general procedure outlined in §2. To carry out this program, it is necessary to make some assumptions about the properties of the basic flow near the axis. We wish to consider smooth flows so that V, W must have Taylor expansions about the origin, but it seems that the only restrictions on the various coefficients we can require generally are that $V(0) = V'(0) = W'(0) = 0$. It emerges that we may need to consider derivatives of V and W up to the fifth and fourth orders respectively and hence that there are a large number of different cases to consider. The same general method is applicable to all of them even though the scaling laws may differ.

We have chosen therefore to illustrate the method by consideration of the trailing vortex (1.4) for which

$$a(r) = \frac{4r^2\,\mathrm{e}^{-r^2}}{(1+\beta^2r^2)^2}\,[q+\beta(\beta q-1)+\beta(1+\beta q)\,r^2+\beta^3r^4], \tag{3.1a}$$

$$b(r) = \frac{4\beta q(1-\beta q)}{1+\beta^2r^2}\,\mathrm{e}^{-r^2}(1-\mathrm{e}^{-r^2}), \tag{3.1b}$$

$$\gamma(r) = n[\beta\,\mathrm{e}^{-r^2}-qr^{-2}(1-\mathrm{e}^{-r^2})]-\omega. \tag{3.1c}$$

In the unstable region discussed in I the value of r_0 is found by insisting that $\gamma'(r_0) = 0$, i.e. it satisfies

$$\mathrm{e}^{r_0^2} = 1+r_0^2+\frac{\beta r_0^4}{q}, \tag{3.2}$$

and r_0 is small when $\beta \approx \frac{1}{2}q$. We shall show that when n is large this is the neighbourhood of the lower neutral point. For more general flows the corresponding requirement is that

$$\beta \approx \lim_{r\to 0}\frac{\Omega'(r)}{W'(r)}. \tag{3.3}$$

We now simplify K by taking r to be small and $\beta = \frac{1}{2}q$ except in the coefficient of r^2 in the expansion of γ. Then

$$K \approx \frac{n^2}{r^2}\left[1+\frac{2q^2(1-\frac{1}{2}q^2)\,r^2}{\gamma^2}+\frac{q(q^2+2)\,r^2}{n\gamma}-\frac{1}{4n^2}+\ldots\right], \tag{3.4}$$

where

$$\gamma = n(\beta-q)-\omega-n(\beta-\tfrac{1}{2}q)\,r^2+\frac{nq}{12}\,r^4+\ldots, \tag{3.5}$$

and the appropriate primary scaling of r and $(\beta-\frac{1}{2}q)$ needed to allow both the bracketed expression in (3.4) and $\gamma'(r)$ to vanish may be seen to be $r \sim n^{-\frac{1}{3}}$ and $\beta-\frac{1}{2}q \sim n^{-\frac{2}{3}}$, with $\gamma \sim n^{-\frac{1}{3}}$. The first two terms in the square bracket in (3.4) are then $O(1)$ while the others are $O(n^{-\frac{4}{3}})$ and may be neglected as long as we ignore contributions to ω that are $O(n^{-\frac{5}{3}})$.

Specifically we adopt the following scaling:

$$\left.\begin{gathered} A = \left[\frac{4\beta}{q}\,(1-\beta q)\right]^{\frac{1}{6}}, \quad r = Azn^{-\frac{1}{3}}, \quad \beta = \tfrac{1}{2}q+qA^2\mu n^{-\frac{2}{3}}, \\ n(\beta-q)-\omega = qA^4\gamma_0\,n^{-\frac{1}{3}}. \end{gathered}\right\} \tag{3.6}$$

Equation (2.2) for ϕ now simplifies to

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}z^2} = \frac{n^2}{z^2}\left(1+\frac{A^2\beta^2z^2}{n^{\frac{2}{3}}}\right)\left[1+\frac{z^2\{1-(\frac{3}{2}+\beta^2)\,A^2z^2n^{-\frac{2}{3}}\}}{\{\gamma_0-\mu z^2+\frac{1}{12}z^4+A^2n^{-\frac{2}{3}}(\frac{1}{2}\mu z^4-\frac{1}{24}z^6)\}^2}+O(n^{-\frac{4}{3}})\right]\phi. \tag{3.7}$$

The eigenvalue problem is to find γ_0 as a function of the real variables μ and A such that ϕ satisfies (3.7), $\phi(0) = 0$ and $\phi \sim z^{-n}$ as $z\to\infty$ along the real axis. The first boundary condition is simply that of smoothness at the axis of the vortex while the second is necessary to ensure a match with the solution when $r = O(1)$ and to be compatible with the central assumption that the dominant behaviour of ϕ occurs in the neighbourhood of the axis.

The general procedure outlined in §2 for $r_0 \neq 0$ may now be used to find γ_0. We look for a value z^* of z at which the coefficient of ϕ in (3.7) has a stationary point. We expand this coefficient in powers of $(z-z^*)$, retaining only the constant and quadratic terms, and note that $z-z^* = O(n^{-\frac{1}{2}})$ if n is to disappear from the leading term of the reduced equation. Finally we determine the value of γ_0 for which this equation has an acceptable eigensolution. The value of z^* for an unstable mode, with $\omega_i > 0$ and $\gamma_{0i} < 0$, is found by requiring that

$$\gamma_0 - \mu z^2 + \tfrac{1}{12}z^4 + \frac{A^2}{n^{\frac{2}{3}}}\left(\tfrac{1}{2}\mu z^4 - \tfrac{1}{24}z^6\right) + \mathrm{i}z\left\{1 - \left(\tfrac{3}{4} + \tfrac{1}{2}\beta^2\right)\frac{A^2 z^2}{n^{\frac{2}{3}}}\right\} \tag{3.8}$$

be stationary and thus, to a first approximation, $z^* = z_0$ where

$$\tfrac{1}{3}z_0^3 - 2\mu z_0 = -\mathrm{i}. \tag{3.9}$$

We have need for γ_0 only to $O(n^{-1})$ and to this order the equation for ϕ reduces to

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}z^2} = \frac{2\mathrm{i}n^2}{z_0^3}\left[\Gamma_0 + \tfrac{1}{2}(z_0^2 - 2\mu)(z-z_0)^2\right]\phi, \tag{3.10}$$

where

$$\Gamma_0 = \gamma_0 - \mu z_0^2 + \tfrac{1}{12}z_0^4 + \mathrm{i}z_0 + \frac{A^2}{n^{\frac{2}{3}}}\left[\tfrac{1}{2}\mu z_0^4 - \tfrac{1}{24}z_0^6 - \mathrm{i}z_0^3\left(\tfrac{3}{4} + \tfrac{1}{2}\beta^2\right)\right]. \tag{3.11}$$

Equation (3.10), in essence, is Weber's equation in the variable Z, where

$$Z = n^{\frac{1}{2}}\left[\mathrm{i}(z_0^2 - 2\mu)\,z_0^{-3}\right]^{\frac{1}{4}}(z-z_0), \tag{3.12}$$

and for most rapid decay of the solution away from z_0, we shall require that Z be real, and seek solutions of (3.10) that are exponentially small as $|Z| \to \infty$. This determines Γ_0 as

$$\Gamma_0 = \frac{\mathrm{i}(2s-1)}{2n}\,z_0^3\left[\mathrm{i}(z_0^2 - 2\mu)\,z_0^{-3}\right]^{\frac{1}{2}}, \tag{3.13}$$

where s ($\geqslant 1$) is an integer. In the z-plane there are two possible directions of integration, differing by $\frac{1}{2}\pi$, through the point z_0. These are given by the fourth roots of $\mathrm{i}(z_0^2 - 2\mu)\,z_0^{-3}$ taken in pairs. Each direction has its associated Γ_0 because the square root in Γ_0 has a different sign for each. The appropriate direction must be determined by a match with the outer solution in the region $|z| = O(1)$. This is a solution of (3.7) for which the path of integration passes through the saddle z_0 and is optimal in the sense described in the Appendix. The appropriate path to choose is the one that tends to infinity in $\mathrm{Re}\,(z) > 0$ in such a way that it does not enclose between itself and the real axis of z a zero of the function γ. We find, and the evidence for this is discussed below, that the required fourth root in (3.12) is that which has argument $\frac{1}{8}\pi$ when z_0 is large, real and positive with the result that γ_0 is determined as

$$\begin{aligned}\gamma_0 &= \tfrac{1}{2}\mu z_0^2 - \tfrac{3}{4}\mathrm{i}z_0 + \frac{A^2}{n^{\frac{2}{3}}}\left[-\tfrac{3}{2}\mu^2 z_0^2 + 3\mathrm{i}\mu z_0\left(\tfrac{3}{2} + \beta^2\right) + 3\left(\tfrac{5}{8} + \tfrac{1}{2}\beta^2\right)\right] \\ &\quad + \mathrm{i}\,\frac{(2s-1)}{2n}\,z_0^3\left[\mathrm{i}(z_0^2 - 2\mu)\,z_0^{-3}\right]^{\frac{1}{2}} + O(n^{-\frac{4}{3}}) \\ &= -\frac{\omega - n(\beta - q)}{qA^4 n^{-\frac{1}{3}}},\end{aligned} \tag{3.14}$$

where the square root in (3.14) has argument $\frac{1}{4}\pi$ when z_0 is large, real and positive.

The last part of (3.14) arises from the definition of γ_0, and is included as a reminder to the reader that this equation fixes ω.

For all real values of μ, of the three roots of (3.9) one is purely imaginary and corresponds to real values of ω so we may reject it. If $\mu > \mu_s$, where

$$\mu_s = -(\tfrac{9}{32})^{\frac{1}{3}} = -0.6552, \tag{3.15}$$

the other two roots are mirror images in the imaginary axis of z, and all roots are imaginary if $\mu < \mu_s$. We choose the root whose real part is positive if $\mu > \mu_s$, in order that the saddle point occurs in $\mathrm{Re}(z) > 0$, and when μ is large and positive we have

$$z_0 = (6\mu)^{\frac{1}{2}} - \frac{\mathrm{i}}{4\mu} + \ldots \tag{3.16}$$

and hence from (3.14) and (3.6) that

$$\omega_1 = \frac{qA^4(6\mu)^{\frac{1}{2}}}{n^{\frac{1}{3}}}\left[1 - \frac{A^2}{n^{\frac{2}{3}}}6\mu(\tfrac{7}{8}+\tfrac{1}{2}\beta^2) - \frac{2s-1}{2n\sqrt{3}}(6\mu)^{\frac{3}{2}} + \ldots\right], \tag{3.17}$$

i.e. $$\omega_1 = [6q(1-\beta q)(2\beta - q)]^{\frac{1}{2}}\left[1 - 6\left(\frac{\beta}{q} - \frac{1}{2}\right)(\tfrac{7}{8}+\tfrac{1}{2}\beta^2) + O(n^{-\frac{1}{3}})\right], \tag{3.18}$$

which agrees with the result in (4.27) of I if, therein, $\beta - \frac{1}{2}q$ is taken to be small. Thus for $\mu \gg 1$ the direction of integration through the saddle is, from (3.12) and our definition of the root, at an angle $-\frac{1}{8}\pi$ to the positive real axis. The other possible direction is at an angle $\frac{3}{8}\pi$ and verification that the correct one has been selected must be made by consideration of the relative positions of the paths of the outer solution and the zeros of the function γ. If $z_{1,4}$ denote the zeros of $\gamma(z)$ lying in the first and fourth quadrants then in the limit $\mu \to \infty$ we have

$$z_{1,4} = (6\mu)^{\frac{1}{2}} \pm \left(\frac{3}{8\mu}\right)^{\frac{1}{4}}(1+\mathrm{i}) \tag{3.19}$$

and it is clear that a path at an angle $-\frac{1}{8}\pi$ may be joined to the origin and to the real z-axis as $|z| \to \infty$ without enclosing either $z_{1,4}$ while the rejected path at angle $\frac{3}{8}\pi$ is in danger of enclosing both z_1 and z_4.

As μ decreases we may trace the properties of z_0, z_1, z_4 and of the angle which the path of integration through the point z_0 must make with the real z-axis. At $\mu = 0$

$$z_0 = 3^{\frac{1}{3}}\,\mathrm{e}^{-\mathrm{i}\pi/6} = 1.249 - 0.721\mathrm{i}, \quad z_1 = 1.833 + 0.491\mathrm{i}, \quad z_4 = 0.491 - 1.833\mathrm{i}. \tag{3.20}$$

The argument of $\mathrm{i}(z_0^2 - 2\mu)z_0^{-3}$ has increased to $\frac{2}{3}\pi$ so the required angle is now $-\frac{1}{6}\pi$. As μ decreases through negative values the argument of $\mathrm{i}(z^2 - 2\mu)z^{-3}$ further increases until it reaches the value $\frac{3}{2}\pi$ at the last admissible value of μ, namely μ_s in (3.15). Here z_0 has a double zero given by

$$z_0 = z_s = -\mathrm{i}(-2\mu_s)^{\frac{1}{2}} = -1.145\mathrm{i}, \tag{3.21}$$

while $$z_1 = 0.780, \quad z_4 = -2.910\mathrm{i}, \quad \gamma_0 = \gamma_{0s} = -0.429 + \ldots \tag{3.22}$$

and the direction of integration is now $-\frac{3}{8}\pi$. Paths for the outer solution are computed in the Appendix together with the positions of the singularities $z_{1,4}$ for $\mu = 0$ to values approaching μ_s. The results given there make clear that the directions of the local integration paths selected here correspond to acceptable paths.

Thus, formally, this method of contour deformation through the saddle has allowed us to reach the lower neutral point. As no singularity of the differential equation is enclosed between the deformed contour and the real axis of z, the eigenvalue ω will have been correctly determined by this procedure though not the properties of the eigenfunction ϕ. The mode separation in ω_i, given by assigning two different values to s in (3.14), is $O(n^{-\frac{4}{5}})$, rather than $O(n^{-\frac{1}{2}})$ as it was in I, and this is in qualitative agreement with the numerical work. It would seem that, from a practical point of view, the reason for any inadequate agreement between ω as derived from (3.14) and as obtained from the integration of the full equation (2.2) in the neighbourhood of $\beta = \frac{1}{2}q$ for large but finite n in the manner carried out in I is mainly due to the replacement of the factor e^{-r^2} in $b(r)$ by unity. At $q = 1$, $\beta = \frac{1}{2}$ and $n = 4$ for instance

$$r_0^2 = 0.413-0.715\,i, \quad e^{-r_0^2} = 0.500+0.434\,i. \tag{3.23}$$

This comparison suggests that for moderate values of n more accurate approximations may be obtained if, in (3.8), the factor $e^{-r^2}(1-e^{-r^2})$ is retained while at the same time $1+\beta^2r^2$ is replaced by unity and the expansion of γ is terminated at r^4. Then (3.8) is replaced by

$$n\beta-nq-\omega-n(\beta-\tfrac{1}{2}q)\,r^2+n(\tfrac{1}{2}\beta-\tfrac{1}{6}q)\,r^4+iA^3q(e^{-r^2}(1-e^{-r^2}))^{\frac{1}{2}} \tag{3.24}$$

and thereafter the same principles are applied as in the determination of (3.14) except that now n is fixed and finite from the outset of the calculations. At $\beta = \frac{1}{2}$, $\mu = 0$ and $n = 4$ this procedure leads to $r_0^2 = 0.39-0.22i$ and $\omega = -1.92+0.34i$ as against the asymptotic result of $\omega = -2.3+0.42i$ and the 'exact' computed value of $-1.853+0.304i$. Another comparison may be made at the neutral point. The requirement for this is that (3.24) should have a point of inflexion on the negative imaginary axis and this leads to $r_0^2 = -0.36$, $\beta = 0.204$ and $\omega = -3.01$. The asymptotic theory gives $r_0^2 = -0.52$, $\beta = 0.24$ and $\omega = -2.56$. It is difficult to pin down the lower neutral point accurately from the calculations. Indeed ω_i remained positive right down to $\beta = 0.2$ when $\omega_i = O(10^{-7})$ but this number is too small to be reliable. The lowest value of β at which it could confidently be claimed that $\omega_i > 0$ is $\beta = 0.222$ where $\omega = -3.003+0.00035i$. Thus, for $n = 4$, the approximate method gives an acceptable estimate for ω. At this value of n the asymptotic formula is less satisfactory; this is not surprising in view of the relatively large values of r and $|\beta-\frac{1}{2}q|$ that occur near the lower neutral point. The assumption that $\frac{1}{2}\beta-\frac{1}{6}q \approx \frac{1}{12}q$ used in (3.11) to obtain the leading term in γ is clearly wrong when $\beta = 0.2$, $q = 1$. None of these remarks invalidates the asymptotic theory; rather they indicate that the structure of the modes at large n is more subtle than might have been thought from the evidence of the numerical results available to date.

4. Approach to the lower neutral point

It is clear that, when $\mu = \mu_s$, the analysis of the previous section fails because the coefficient of $(z-z_0)^2$ in (3.10) vanishes. A further refinement of the scaling is necessary in the neighbourhood of $\mu = \mu_s$ to overcome this deficiency and this also enables us to separate the modes to leading order in the differential equation.

We write

$$\mu = \mu_s+\epsilon \tag{4.1}$$

where ϵ is a small positive number, so that

$$z_0 = z_s+(2\epsilon)^{\frac{1}{2}}+\dots \tag{4.2}$$

and

$$\Gamma_0 = \gamma_0-\gamma_{0s}-2\epsilon\mu_s+\tfrac{8}{3}i(-\mu_s)^{\frac{1}{2}}\epsilon^{\frac{3}{2}}+\frac{3i}{4}\frac{A^2\epsilon^{\frac{1}{2}}}{n^{\frac{2}{3}}}(3+q^2)\,(2\mu_s^2)^{\frac{1}{2}}+\dots, \tag{4.3}$$

where, as in (3.22), γ_{0s} is the value of γ_0 as given by (3.14), evaluated at $\mu = \mu_s$ and $z = z_s$. Then ϕ satisfies

$$\frac{d^2\phi}{dz^2} = \frac{4n^2}{3}[\Gamma_0 - 2i(-\epsilon\mu_s)^{\frac{1}{2}}(z-z_0)^2 - \tfrac{1}{3}i(-2\mu_s)^{\frac{1}{2}}(z-z_0)^3 + \dots]\phi \tag{4.4}$$

in the neighbourhood of $z = z_0$. The equation may be reduced to a simplified form in which n is explicitly absent by writing

$$\Gamma_0 = \left(\frac{-9\epsilon\mu_s}{4}\right)^{\frac{1}{4}} e^{3i\pi/4}\frac{\sigma}{n}, \quad \tau = \frac{1}{6}\left(\frac{-4\epsilon^5\mu_s}{9}\right)^{-\frac{1}{8}}\frac{e^{-3i\pi/8}}{n^{\frac{1}{2}}}$$

and setting

$$z = z_0 + \left(\frac{-64\epsilon\mu_s}{9}\right)^{-\frac{1}{8}}\frac{e^{-3i\pi/8}\eta}{n^{\frac{1}{2}}}, \tag{4.5}$$

where $|\eta| = O(1)$. The form it takes is

$$\frac{d^2\phi}{d\eta^2} = (\sigma + \eta^2 + \tau\eta^3)\phi, \tag{4.6}$$

and as usual we must require that $\phi \to 0$ as $|\eta| \to \infty$ along the real axis of η. As in §3 there are two possible directions of the path through z_0, that in (4.5) being chosen because we know from §3 that as $\mu \to \mu_s$ the limiting path direction makes an angle $-\frac{3}{8}\pi$ with the real axis of z. When $|\tau|$ is small we may write ϕ and σ as power series in τ of which the leading terms are respectively Weber functions and $-(2s-1)$. Further terms may be computed by successive substitution, the details being similar to those in I. For the first two modes

$$\sigma_1 = -1 + \tfrac{11}{16}\tau^2 + 1.816\tau^4 + O(\tau^6), \tag{4.7}$$

$$\sigma_2 = -3 + 4.438\tau^2 + 21.97\tau^4 + O(\tau^6), \tag{4.8}$$

so that they are now well separated when τ is small and the difficulties experienced with mode-jumping in earlier computations are easier to avoid.

For larger values of $|\tau|$, a numerical solution of the eigenvalue problem (4.6) with $\phi \to 0$ as $|\eta| \to \infty$ is required. This has been carried out, shooting with a fourth- and fifth-order Runge–Kutta algorithm. Integrations were started from $\eta = -L$, and terminated at $\eta = L$, where $L = 4$ was found sufficient for all values of $|\tau|$. At $\eta = -L$, an asymptotic condition (which may be found by WKB analysis) appropriate for $|\eta| \to \infty$,

$$\frac{\phi_\eta}{\phi} \sim \eta[1 + \tau\eta]^{\frac{1}{2}} \tag{4.9}$$

was used, with $\phi(-L)$ set at a small value (10^{-5} or 10^{-6}). The boundary condition as $\eta \to \infty$ is replaced by the asymptotic condition (4.9) and Muller's method is used to adjust σ so that (4.9) is satisfied at $\eta = L$.

The results of this numerical study are presented in figure 1. As $|\tau| \to \infty$, $|\sigma| \to \infty$. The limit problem is easily solved by the same methods after the transformation

$$\eta = |\tau|^{-\frac{1}{5}} Y, \quad \sigma = |\tau|^{\frac{2}{5}} \lambda. \tag{4.10}$$

The limiting value of λ is found to be

$$\lambda = -(1.1420 + 0.1809i). \tag{4.11}$$

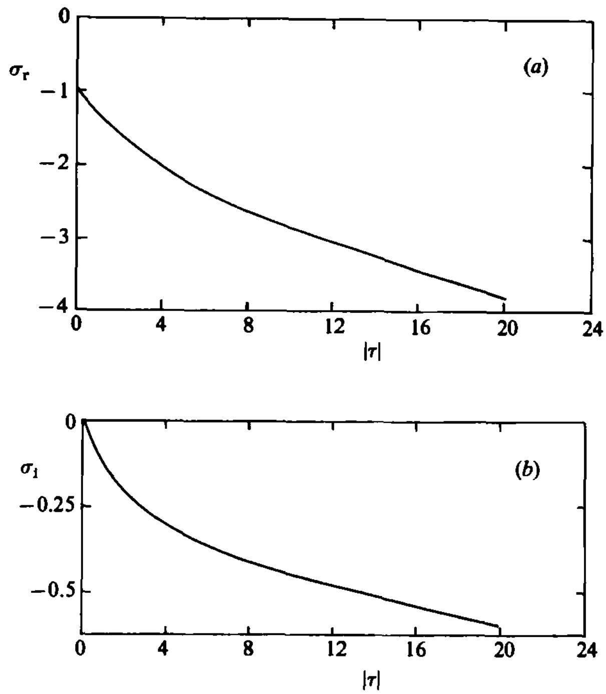

FIGURE 1. Real (*a*) and imaginary (*b*) parts of the eigenvalue σ of (4.6) as functions of $|\tau|$. The asymptotic behaviour of σ is given by $\sigma = -|\tau|^{\frac{8}{5}}(1.1420+0.1809\mathrm{i})$.

We now wish to determine whether neutral modes are captured by the asymptotic analysis in this section. Now

$$\frac{\omega - n(\beta - q)}{qA^4 n^{-\frac{1}{3}}} = -\gamma_0$$

$$= -\Gamma_0 - \gamma_{0\mathrm{s}} - 2\mu_\mathrm{s}\,\epsilon + \tfrac{8}{3}\mathrm{i}\epsilon^{\frac{3}{2}}(-\mu_\mathrm{s})^{\frac{1}{2}} + \tfrac{3}{4}\mathrm{i}(3+4\beta^2)\,A^2 n^{-\frac{2}{3}}(2\epsilon\mu_\mathrm{s}^2)^{\frac{1}{2}},$$

and the imaginary part of $\gamma_{0\mathrm{s}}$ is zero. Thus

$$\omega_\mathrm{i} = qA^4 n^{-\frac{1}{3}}\Big\{\tfrac{8}{3} - (\mu_\mathrm{s})^{\frac{1}{2}}\,\epsilon^{\frac{3}{2}} + \tfrac{3}{4}(3+4\beta^2)\,(2\mu_\mathrm{s}^2)^{\frac{1}{2}}\,A^2\epsilon^{\frac{1}{2}}n^{-\frac{2}{3}}$$

$$+\,\epsilon^{\frac{1}{4}}n^{-1}\left(-\frac{9\mu_\mathrm{s}}{4}\right)^{\frac{1}{4}}\mathrm{Im}\,(-\sigma\,\mathrm{e}^{3\pi\mathrm{i}/4})\Big\}. \quad (4.12)$$

The coefficients of the $\epsilon^{\frac{3}{2}}$ and $\epsilon^{\frac{1}{2}}n^{-\frac{2}{3}}$ terms are both positive, so that neutral conditions may be reached at non-zero values of ϵ only if the third term cancels the remaining terms within the braces. This last term, however, is positive, since $\mathrm{Im}\,(-\sigma\,\mathrm{e}^{3\pi\mathrm{i}/4}) = (\sigma_\mathrm{i} - \sigma_\mathrm{r})/\sqrt{2}$, and this is found to be positive for all $|\tau|$.

Formally, (4.12) gives $\omega_\mathrm{i} = 0$ when $\epsilon = 0$, but this corresponds to $|\tau| = \infty$, since $|\tau| \propto \epsilon^{-\frac{5}{8}}n^{-\frac{1}{2}}$. But the asymptotic behaviour of σ as $|\tau| \to \infty$ implies that the third term inside the braces in (4.12) is proportional to $n^{-\frac{6}{5}}(\lambda_\mathrm{i} - \lambda_\mathrm{r})$ with a positive constant of proportionality, as $|\tau| \to \infty$. It therefore does not vanish as $\epsilon \to 0$, so that the formal indication of neutral conditions is incorrect. Furthermore, (4.4) is valid provided $\epsilon \gg n^{-\frac{4}{3}}$; if ϵ is $O(n^{-\frac{4}{3}})$ or smaller, the coefficient of $(z - z_0)^2$ in (4.4) is altered. Thus, it is not valid to take the limit $\epsilon \to 0$ in (4.12).

Thus, the approach to the neutral point must be resolved by additional analysis in a disc within ϵ of $O(n^{-\frac{4}{3}})$ of the point $z = z_s$; this is not attempted here.

5. Marginal stability

The theory as developed so far requires that $1-\beta q > 0$, in order that $b(r) > 0$, and also $\beta > \frac{1}{2}q$ for the existence of gross instabilities as described in I while $\beta-\frac{1}{2}q = O(n^{-\frac{2}{3}})$ for the weaker instabilities of §§3, 4. Thus if $q < \surd 2$ the columnar vortex is certainly unstable and if $q > \surd 2$ all the unstable modes that we have discussed here disappear as $n \rightarrow \infty$. The condition $q < \surd 2$ is not in fact necessary for instability. An estimate of the range of q beyond $\surd 2$ at which instabilities can occur when $n \gg 1$ but finite may be obtained by an extension of the theory of §3. We write

$$q = \surd 2\left(1+\frac{Q}{n}\right), \quad \beta = \frac{1}{\surd 2}\left(1-\frac{P}{n}\right), \tag{5.1}$$

where Q, P are both $O(1)$ with $P > Q$ (since $\beta q < 1$) and n is large. Then by use of arguments similar to those of §3 we write

$$\left.\begin{aligned} r = \frac{R}{n^{\frac{1}{2}}}[2(P-Q)]^{\frac{1}{6}}, \quad P+Q = -2\nu[2(P-Q)]^{\frac{1}{3}}, \\ n(\beta-q)-\omega = \frac{\surd 2}{n}[(P-Q)]^{\frac{2}{3}}\gamma_{\mathrm{M}}, \end{aligned}\right\} \tag{5.2}$$

and the equation for ϕ reduces to

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}R^2} = \frac{n^2}{R^2}\left[1+\frac{R^2}{(\frac{1}{12}R^4-\nu R^2+\gamma_{\mathrm{M}})^2}+O\left(\frac{1}{n}\right)\right]\phi. \tag{5.3}$$

Thus the theory for the marginal stability state is similar to that for the neighbourhood of the lower neutral point except for the implicit scaling and the error term in (5.3). The correction due to the non-zero value of z_0 is now of the same order of magnitude as the mode-separation term, i.e. gives a correction to ω that is $O(n^{-2})$. Of special interest is the largest possible value of Q at which a neutral mode is possible. This may be found by requiring that

$$-\nu = \frac{P+Q}{2[2(P-Q)]^{\frac{1}{3}}} \tag{5.4}$$

has a minimum value in $P > Q$ of $-\mu_s = (\frac{9}{32})^{\frac{1}{3}}$. The minimum value occurs at $P = 2Q$ and is $3(\frac{1}{4}Q)^{\frac{2}{3}}$. Hence the vortex is stable for large enough n if

$$Q > \frac{1}{\surd 6}. \tag{5.5}$$

For smaller positive values of Q the range of admissible P is $Q < P_1 \leqslant P \leqslant P_2 < \infty$ where P_1, P_2 are the values of P for which $\nu = -0.655$. Both neutral points are therefore of the type hitherto described as lower neutral points and the theory of §4 is applicable. However, if $Q < 0$, $(q < \surd 2)$, $P_1 = Q$ and ν is unbounded from above. The properties of the upper neutral point $(P = Q)$ then require a theory of the kind described by Stewartson & Capell (1985) in which the computations and analysis of I are extended to the neighbourhood of $\beta = q^{-1}$. The theory of §4 still applies to the neutral point P_2.

q	$\alpha = n\beta$	$\omega - \alpha + nq$	μ
1.36	2.48	0.145+0.096i	−0.169
1.40	2.44	0.139+0.060i	−0.256
1.44	2.40	0.123+0.029i	−0.344
1.46	2.38	0.112+0.016i	−0.390
1.48	2.36	0.044+0.006i	−0.435
1.50	2.34	0.021+0.001i	−0.481
1.52	2.22	0.066+0.001i	−0.556
1.54	2.18	0.023+0.0001i	−0.603
1.56	2.21	0.070+0.0001i	−0.633

TABLE 1. Principal properties of unstable mode for $q \geqslant 1.36$, $n = 4$

q	$\alpha = n\beta$	$\omega - \alpha + nq$	μ
1.40	3.13	0.129+0.063i	−0.256
1.42	3.11	0.122+0.044i	−0.308
1.44	3.08	0.113+0.027i	−0.365
1.46	3.04	0.103+0.014i	−0.427
1.48	2.98	0.093+0.004i	−0.495
1.50	2.95	0.079+0.001i	−0.551
1.52	2.90	0.055+0.0001i	−0.612
1.54	2.85	0.032+0.0003i	−0.671

TABLE 2. Principal properties of unstable mode for $q \geqslant 1.40$, $n = 5$

Numerical computations of the unstable modes, using the methods described in I, are very expensive and time-consuming especially when $q > \sqrt{2}$ and n is small. It was possible to carry out only a limited study when $n = 4, 5$ and special attention was paid to the modes with maximum growth rate for fixed q as α varies. These modes, which are believed to correspond to $s = 1$, are displayed in tables 1 and 2, together with the appropriate value of μ which should be approaching -0.6552 at the neutral point. For $q < \sqrt{2}$ the overall agreement between the numerical and asymptotic estimations of ω is only moderately encouraging. On the specific question of the largest value of q that permits instability as α varies for fixed n the agreement is good. Thus (5.5) predicts values of α and q of 2.251 and 1.558 at $n = 4$, and of 2.958 and 1.530 at $n = 5$.

6. Discussion

The procedure, first given in I, for obtaining the asymptotic expansion of the eigenvalue ω when $n \gg 1$ for finite values of β has been extended in this paper to provide further information about the properties of ω near the lower neutral point. The strategy of this method is to find a point r_0 in the complex r-plane at which K in (2.5) is stationary. The procedures developed here, as in our previous paper I, are worked out in detail for the basic flow (1.4), but are clearly applicable to a wide class of flows.

In I, flow (1.4) was considered for (reduced) wavenumbers β in the range $\frac{1}{2}q < \beta < q^{-1}$; r_0 is then real to a first approximation. In that case the eigenfunction was expanded as a power series in $n^{-\frac{1}{4}}$ in a neighbourhood of r_0 with size of $O(n^{-\frac{3}{4}})$

and ω was obtained as a power series in $n^{-\frac{1}{2}}$. The lower neutral point is then correctly predicted as $\beta = \frac{1}{2}q$ in the limit $n \rightarrow \infty$ but if $n = 4$, $q = 1$, the numerical result is $\beta \approx 0.22$. This poor agreement contrasts sharply with the excellent correlation for the maximum growth rate reported in I.

We have shown here that, when $\beta - \frac{1}{2}q = O(n^{-\frac{2}{3}})$, the necessary modification to the above approach is to expand ω as a power series in $n^{-\frac{1}{3}}$ and the eigenfunction as a power series in $n^{-\frac{1}{6}}$ in a neighbourhood of r_0 with size of $O(n^{-\frac{5}{6}})$. This stationary point r_0 of K is now complex and such that $|r_0| = O(n^{-\frac{1}{3}})$. The deformation of the contour of integration to go through this point does not have to pass through any singular points of the differential equation as long as the assumption of a positive (or negative) value of ω_1 is consistent. It emerges that generally the comparison with the numerical results is less favourable than previously, the most likely reason being that only the leading term in the expansion is considered here while four were computed in I. For example, if the leading term only is retained in I then, at $q = 1$, $n = 4$, $\beta = 0.574$ for the primary mode, $\omega = -1.725 + 0.405\mathrm{i}$ as against the computed value $\omega = -1.681 + 0.339\mathrm{i}$, and the asymptotic result, accounting for four terms in the expansion, of $\omega = -1.646 + 0.346\mathrm{i}$. These discrepancies between the leading term and the numerical results found in I are comparable with those found here and the inclusion of higher-order terms would presumably result in an improvement in accuracy comparable with that found in I. Our principal goal here is to estimate the position of the neutral point when n is large but finite and the results given are in decidedly better agreement with the behaviour given by the numerical computations, even to the extent of placing an upper bound on q for instability for n greater than about 4 or 5.

The computer runs carried out in I to determine ω proved to be extremely expensive, very small step sizes being required not only in r but also in the hunting process whereby the behaviour of ω is traced as a function of q, β for fixed n. Considerable use of the technique of contour deformation was made but even so these difficulties, and the ever-present danger of mode-jumping, effectively prevented us from extending these computations to neutral conditions. The analytical theory of this paper provides a clue to the resolution of these difficulties, for near the neutral mode the critical value r_0 of r actually lies on the negative imaginary axis of r and the choice of contour in the theory, made to ensure that K is real and has a minimum at r_0, requires that it pass from the third quadrant into the fourth quadrant in such a direction that the real part of r is increasing while the imaginary part is decreasing. Thus the best contour from the point of view of the computations may be one that starts at $r = 0$ and then allows the real part of r to *decrease* before turning round and ending at $r = \infty - 0\mathrm{i}$. The choice of such a contour at finite values of n is clearly a formidable task, but the asymptotic theory may help to provide a convenient start.

The asymptotic theory also throws some light on the phenomenon of mode-jumping, for it shows that as

$$\mu = \frac{\beta - \frac{1}{2}q}{q}\left[\frac{n^2}{2-q^2}\right]^{\frac{1}{3}} \tag{6.1}$$

decreases to negative values the modes, defined by assigning different integer values to s in (3.14), come closer together and eventually differ by $O(n^{-\frac{5}{3}})$ very near the lower neutral point where $\mu = -0.6552$. Again a choice of contour to take advantage of the present theory might be helpful in the computation of the leading modes near the lower neutral points.

The first drafts of this paper were prepared by Keith Stewartson shortly before his death in May 1983. Dr Susan Brown gave generously of her time in reading, checking, and correcting the paper, and I am most grateful to her. Responsibility for any errors which may remain is mine. The work was supported in part by Grant MEA 83-06713 of the NSF Fluid Mechanics Program and in part by contract ONR SRO-IV with the US Office of Naval Research.

Appendix

Computation of contour deformation leading to steepest descent outer paths is described here. These are calculated for the scaled problem (3.7), with terms proportional to $n^{-\frac{2}{3}}$ neglected. Then

$$\frac{d^2\phi}{dz^2} = n^2 G(z)\,\phi, \tag{A 1}$$

where

$$G(z) = \frac{1}{z^2}\left\{1+\frac{z^2}{(\gamma_0-\mu z^2+\frac{1}{12}z^4)^2}\right\} = |G|\,e^{i\psi}.$$

Introduce a real parameter t, and let $z = z(t)$ be the contour to be determined in the complex z-plane. Then, to lowest order (A 1) is

$$\ddot{\phi} = n^2\dot{z}^2|G|\,e^{i\psi}\,\phi, \tag{A 2}$$

where dots signify differentiation with respect to the real variable t. If $\dot{z}^2\exp(i\psi)$ is real and positive everywhere on a path passing through z_0, then on such a path ϕ will decay (or grow) exponentially fast, and the choice of such a steepest descent path will lead to the least correction to the local analysis of §2. There is no guarantee that such a path can be made to pass through the boundary points $z = 0$ and $z = \infty$, but the steepest descent paths can be linked to the boundary points by other path segments with errors that have been made exponentially small by travelling as far as possible on the steepest descent paths. We compute paths by requiring $z(t)$ to satisfy

$$\dot{z}^2\,e^{i\psi} = 1, \tag{A 3}$$

with $z(0) = z_0$, the stationary point found earlier. At $z = z_0$, G has a double zero, so near $z = z_0$,

$$G = c(z-z_0)^2, \quad c = \frac{i(z_0^2-2\mu)}{z_0^3}.$$

Set $c = |c|\,e^{i\delta_1}$. Near $t = 0$, $z = z_t + t\,e^{i\delta_2}$, and

$$G = |c|\,t^2\,e^{2i\delta_2+i\delta_1}.$$

Thus, to satisfy (A 3),

$$\dot{z}(0) = e^{-i\delta_1/4}\,e^{2\pi i p/4}, \quad p = 0,1,2,3.$$

Choice of p gives the four possible directions of steepest descent rays from $z = z_0$: These come in $(\pm)$ pairs, determining two lines which intersect at right angles at z_0. To compute the 'optimal' paths, a single step away from z_0 was taken using Euler's

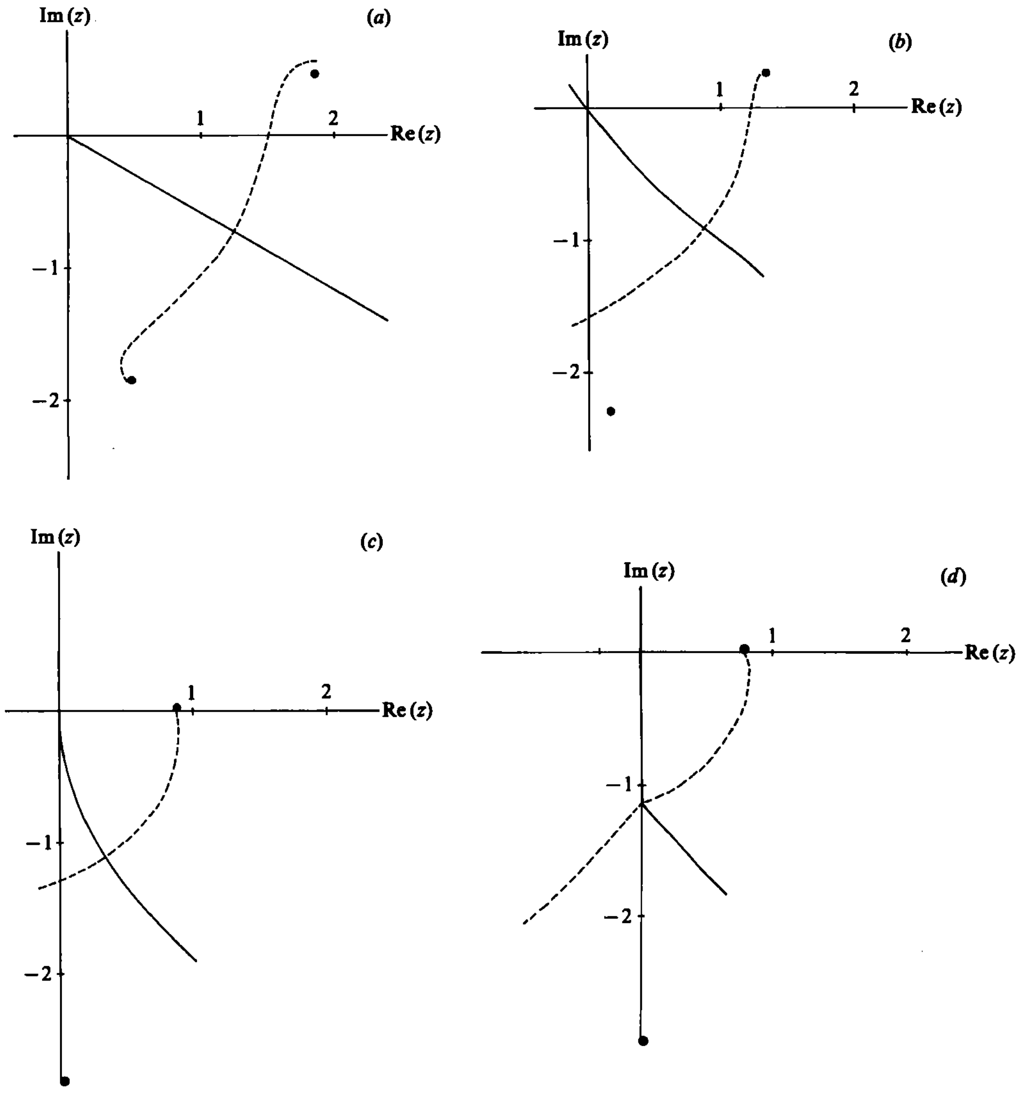

FIGURE 2. Outer integration paths in the complex z-plane, with positions of the singularities of K marked by the dots. An acceptable path, which must not have a singularity between it and the real z-axis, is drawn as a solid curve. The other, unacceptable, path is drawn dashed. The intersection of the solid and dashed curves is the stationary point z_0. (*a*) $\mu = 0$, (*b*) $\mu = -0.3$, (*c*) $\mu = -0.6$, (*d*) $\mu = -0.655$. Note the incipient development of a kink in the acceptable path.

algorithm along one of the four descent directions. From the point so reached, a fourth-order Runge–Kutta formula was used to continue the integration of

$$\dot{z}(t) = \pm e^{-i\psi(z)/2},$$

the appropriate branch ($\pm$) being selected.

The results of these computations are shown in figure 2 for four values of μ approaching the limiting value $\mu_s = -0.6552-$. We note that in each case, one path through z_0 is acceptable (drawn unbroken) as it can be obtained by deforming the contour from the real z-axis without passing over a singularity of G, or equivalently, of K (those in the first and fourth quadrant are marked on the figures), while the second path (drawn with a broken line) is not acceptable.

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